RULE 1 **Derivative of a Constant Function**

$$f(x) = c$$

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If f is the function with the constant value c, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Derivative of a Power Exploration

RULE 2 Power Rule for Positive Integer Powers of x

$$f(x) = x^n$$

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

$$a = x + h \text{ and } b = x.$$

RULE 2 Power Rule for Positive Integer Powers of x

If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

RULE 3 The Constant Multiple Rule

If u is a differentiable function of x and c is a constant,

$$f(x) = cu(x)$$

RULE 3 The Constant Multiple Rule

If u is a differentiable function of x and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

RULE 4 The Sum and Difference Rule

If u and v are differentiable functions of x, then their sum and difference are differentiable at every point where u and v are differentiable. At such points,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}.$$

EXAMPLE 1 **Differentiating a Polynomial**

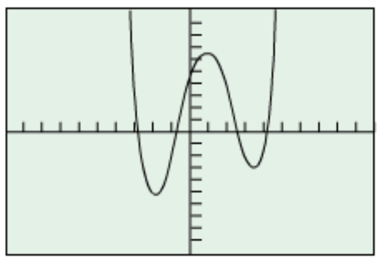
Find $\dfrac{dp}{dt}$ if $p = t^3 + 6t^2 - \dfrac{5}{3}t + 16.$

EXAMPLE 2 Finding Horizontal Tangents

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

EXAMPLE 3 Using Calculus and Calculator

As can be seen in the viewing window $[-10, 10]$ by $[-10, 10]$, the graph of $y = 0.2x^4 - 0.7x^3 - 2x^2 + 5x + 4$ has three horizontal tangents (Figure 3.18). At what points do these horizontal tangents occur?



[–10, 10] by [–10, 10]

Derivative of a Product Exploration

RULE 5 The Product Rule

$$\frac{d}{dx}uv$$ where u is u(x) and v is v(x)

To change the fraction into an equivalent one that contains difference quotients for the derivatives of u and v, we subtract and add $u(x+h)v(x)$ in the numerator. Then,

$$\frac{d}{dx}(uv) = \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}$$

EXAMPLE 4 Differentiating a Product

Find $f'(x)$ if $f(x) = (x^2 + 1)(x^3 + 3)$.

Derivative of a Quotient Exploration

$$f(x) = \frac{x^3}{\sin x}$$

1. $f'(1) \approx 2.802$

2. $\frac{3x^2}{\cos x}$ $\Downarrow$ not $f'(x)$ $\qquad \frac{3(1)^2}{\cos 1} \approx 5.552$

not = to the numerical derivative

3. $y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$

4. $\lim_{\Delta x \to 0} \frac{\frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}}{\Delta x}$

5. $\frac{1}{\Delta x} \cdot \frac{v(u+\Delta u) - u(v+\Delta v)}{v(v+\Delta v)}$

$\frac{1}{\Delta x} \cdot \frac{\cancel{vu} + v\Delta u - \cancel{vu} - u\Delta v}{v(v+\Delta v)}$

$\frac{1}{\Delta x} \cdot \frac{v\Delta u - u\Delta v}{v(v+\Delta v)}$

6. $\lim_{\Delta x \to 0} \frac{v\left(\frac{\Delta u}{\Delta x}\right) - u\left(\frac{\Delta v}{\Delta x}\right)}{v(v+\Delta v)}$

$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \to$ derivative of u $\qquad \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \to$ derivative of v

7. $\frac{v(u') - u(v')}{v^2}$

8. $v : \sin x \quad u : x^3$

$$\frac{\sin x(3x^2) - x^3(\cos x)}{\sin^2 x}$$

$$\frac{\sin(1)(3(1)^2) - 1^3(\cos(1))}{(\sin(1))^2}$$

≈ 2.802

bottom times the derivative of the top minus top times the derivative of the bottom all divided by the bottom squared.

RULE 6 The Quotient Rule

At a point where $v \neq 0$, the quotient $y = u/v$ of two differentiable functions is differentiable, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

EXAMPLE 5 Supporting Computations Graphically

Differentiate $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$. Support graphically.

$u = x^2 - 1$ $\quad$ $v = x^2 + 1$

$u' = 2x$ $\quad$ $v' = 2x$

$$= \frac{(x^2+1)(2x) - [(x^2-1)(2x)]}{(x^2+1)^2}$$

$$= \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2+1)^2}$$

$$= \frac{4x}{(x^2+1)^2}$$

EXAMPLE 6 Working with Numerical Values

Let $y = uv$ be the product of the functions u and v. Find $y'(2)$ if

$$u(2) = 3, \quad u'(2) = -4, \quad v(2) = 1, \quad \text{and} \quad v'(2) = 2.$$

if $y = \frac{u}{v}$

$$= \frac{v(u') - u(v')}{v^2}$$

$$= \frac{1(-4) - 3(2)}{1^2}$$

$$= \frac{-4-6}{1}$$

$$= -10$$

if $y = uv$

$$= u(v') + v(u')$$

$$= 3(2) + 1(-4)$$

$$= 6 + (-4)$$

$$= 2$$

EXAMPLE 7 Using the Power Rule

Find an equation for the line tangent to the curve

$$y = \frac{x^2 + 3}{2x}$$

at the point (1, 2). Support your answer graphically.

slope → derivative when $x = 1$

point $(1, 2)$

$$y' = \frac{2x(2x) - (x^2+3)(2)}{(2x)^2}$$

$$= \frac{4x^2 - 2x^2 - 6}{4x^2}$$

$$y'(1) = \frac{2(1)^2 - 6}{4(1)^2}$$

$$= \frac{-4}{4} = -1$$ → slope of tangent

$$y - 2 = -(x - 1)$$

Second and Higher Order Derivatives

The derivative $y' = dy/dx$ is called the *first derivative* of y with respect to x. The first derivative may itself be a differentiable function of x. If so, its derivative,

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2},$$

is called the *second derivative* of y with respect to x. If y'' ("y double-prime") is differentiable, its derivative,

$$y''' = \frac{dy''}{dx} = \frac{d^3y}{dx^3},$$

is called the *third derivative* of y with respect to x. The names continue as you might expect they would, except that the multiple-prime notation begins to lose its usefulness after about three primes. We use

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} \qquad \text{"}y \text{ super } n\text{"}$$

to denote the ***n*th derivative** of y with respect to x. (We also use $d^n y/dx^n$.) Do not confuse $y^{(n)}$ with the nth power of y, which is y^n.

EXAMPLE 8 **Finding Higher Order Derivatives**

Find the first four derivatives of $y = x^3 - 5x^2 + 2$.

$y' = 3x^2 - 10x$

$y'' = 6x - 10$

$y''' = 6$

$y'''' = 0$

EXAMPLE 9 Finding Instantaneous Rate of Change

An orange farmer currently has 200 trees yielding an average of 15 bushels of oranges per tree. She is expanding her farm at the rate of 15 trees per year, while improved husbandry is improving her average annual yield by 1.2 bushels per tree. What is the current (instantaneous) rate of increase of her total annual production of oranges?

Let the functions t and y be defined as follows.

$t(x)$ = the number of trees x years from now. trees/yr

$y(x)$ = yield per tree x years from now. bushels of oranges/tree

$$O(x) = t(x) \cdot y(x)$$

$$O'(x) = t(x) \cdot y'(x) + y(x) \cdot (t'(x))$$

$t(0) = 200$ $\quad t'(0) = 15$

$y(0) = 15$ $\quad y'(0) = 1.2$

$$O'(x) = 200(1.2) + 15(15)$$

$$= 465 \text{ bushels of oranges / year}$$